Vital Annex: International Journal of Novel Research in Advanced Sciences (IJNRAS)
Volume: 02 Issue: 07 | 2023 ISSN: 2751-756X
http://innosci.org

Optimal Cubature Formulas on a Sphere of Three-Dimensional Space

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Annotation: This article shows the construction of an optimal cubic formula for the integration of functions defined on the unit sphere. By transferring the functions defined on the unit sphere to the spherical coordinate system, multiple integrals are made, and optimal quadrature formulas for each variable are constructed.

Keywords: optimal quadrature formula, optimal cubic formula, unit sphere.

Introduction. In this work, we consider the issue of constructing cubic formulas with optimal quadrature formulas for integrating functions defined on the unit sphere. For this purpose, we present the following definitions and formulas known in advance.

$L_2^{(m)}(0,1)$ – the space of Sobolev functions that are absolutely continuous up to a derivative of order (m–1) and a generalized derivative of m-order are integrated with a square. In the space $L_2^{(m)}(0,1)$, the scalar product of the elements ψ and φ is defined as follows [1]

$$<\varphi,\psi>=\int_0^1 \psi^{(m)}(x)\varphi^{(m)}(x)dx. \qquad (1)$$

In this case, the norm corresponding to the scalar product (1) of the function $\varphi\in L_2^{(m)}(0,1)$ is defined in this form

$$\|\varphi\|_{L_2^{(m)}(0,1)}=\left[\int_0^1 (\varphi^{(m)}(x))^2dx\right]^{\frac{1}{2}}. \qquad (2)$$

Let $\mathbf{R}^3$ be a Euclidean space, the scalar product of two vectors $x=(x_1, x_2, x_3)^T$ and $y=(y_1, y_2, y_3)^T$ in three-dimensional space is defined by the equality $x\cdot y=x_1\cdot y_1+x_2\cdot y_2+x_3\cdot y_3$, and the norm of the element x in the space $\mathbf{R}^3$ is as follows (see [2])

$$\|x\|_2=\sqrt{x_1^2+x_2^2+x_3^2}.$$

The spherical coordinate system is convenient to use in the mathematical solution of practical problems related to the sphere. In this case, the transition from Cartesian to spherical coordinates is carried out according to the formulas:

$$x=r\sin\theta\cos\varphi,\ \ y=r\sin\theta\sin\varphi,\ z=r\cos\theta,$$

where $r>0$, $0\le\varphi<2\pi$ and $0\le\theta\le\pi$.

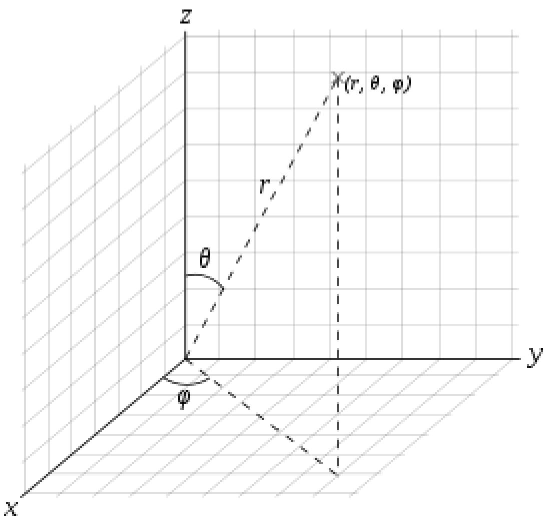

Fig 1. In a spherical coordinate system, θ is the zenith and φ azimuth angles, r is the distance from the origin of coordinates to a given point.

The set of all elements $\xi\in\mathbf{R}^3$ for which the condition is satisfied is called the unit sphere and is denoted by $\mathbf{S}^2$, i.e.

$$\mathbb{S}^2 = \left\{ \xi \mid \xi \in \mathbb{R}^3,\ \|\xi\|_2 = 1 \right\},$$

where $\xi = (\xi_1, \xi, \xi_3)$. All points of the unit sphere with spherical coordinates can be expressed as

$$\xi = \xi\left(\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta\right), \qquad (3)$$

where θ zenith and φ azimuth angles $0 \le \theta \le \pi$ and $0 \le \varphi < 2\pi$ satisfy the above conditions, respectively (Fig. 1)

We define the class of continuous functions defined on the sphere $\mathbf{S}^2$ as $C(\mathbf{S}^2)$ and the norm of a function in this class of functions is given as follows

$$\| f \|_{C(\mathrm{S}^2)} = \sup_{\xi\in \mathrm{S}^2} | f(\xi) |.$$

The space of square-integrable functions on the unit sphere $\mathbf{S}^2$ is denoted by $L_2(\mathbf{S}^2)$. The norm of a function f that belongs to the space $L_2(\mathbf{S}^2)$ defined on the sphere $\mathbf{S}^2$ is defined as follows

$$\| f \|_{L_2(\mathrm{S}^2)} = \left(\int_{\mathrm{S}^2} | f(\xi) |^2 \, d\mathrm{S}^2(\xi) \right)^{\frac{1}{2}}.$$

Here $d\mathbf{S}^2(\xi)$ (in the sense of Lebesgue) is an element of the sphere $\mathbf{S}^2$, $\xi = (\xi_1, \xi, \xi_3)\in\mathbf{S}^2$. $L_2(\mathbf{S}^2)$ is a Hilbert space in which the inner product between the elements f and g is introduced as (see, for example, [2])

$$(f, g)_{L_2(\mathrm{S}^2)} = \int_{\mathrm{S}^2} f(\xi) g(\xi) d\mathrm{S}^2(\xi).$$

With numerical integration of the function $f \in L_2(\mathbf{S}^2)$ defined on the unit sphere, in the spherical coordinate system, the calculation formula has the form

$$\int_{S^2} f(\xi_1,\xi_2,\xi_3)dS^2 = \int_0^{2\pi}\int_0^{\pi} f(\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta)|J|d\theta d\varphi,$$

where $|J| = \sin\theta$ is the Jacobian of the transformation. If we denote the integrand on the right side of the last equality by $F(\theta, \varphi)$, i.e.

$f(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)=F(\theta, \varphi)$,

we get the following:

$$\int_0^{2\pi}\int_0^{\pi} f(\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta)|J|d\theta d\varphi = \int_0^{2\pi}\int_0^{\pi} F(\theta,\varphi)\sin\theta d\theta d\varphi. \quad (4)$$

When calculating the integral (4) approximately, given that the function $F(\theta, \varphi)$ is periodic in φ, the following cases can be considered respectively, depending on whether it is periodic or non-periodic in θ.

1) Suppose that the function $F(\theta, \varphi)$ with the integral (4) is a periodic function for the variables θ and φ, here we can use the optimal quadrature formula for periodic functions with weight $p(x)$ in the variable θ [1]

$$\int_0^1 p(x)\varphi(x)dx \cong \sum_{\beta=1}^{N} C[\beta]\varphi[\beta].$$

Then we use the following generalized rectangle formula for the second variable φ

$$\int_a^b f(x)dx \cong \sum_{\beta=0}^{N} hf(a+h\beta), \quad (5)$$

where $h = \frac{b-a}{N}$, N – natural number [3].

2) If the function $F(\theta, \varphi)$ with the integral (4) is not periodic in the variable θ, then the optimal quadrature formula can be used for the next weight of the sine for this variable [4]

$$\int_0^{\pi} \sin(\theta)F(\theta,\varphi)d\theta \cong \sum_{\beta=0}^{N} C[\beta]F(h\beta,\varphi). \quad (6)$$

Suppose that the integrand in the integral (4) is periodic in the variable φ for a given function $F(\theta, \varphi)$, then the line generalizing the approximate integral (6) in the variable φ ($0 \le \varphi < 2\pi$) can be calculated using the quadrilateral formula. As a result, we will have an optimal cubic formula for the approximate calculation of integrals of functions defined on a sphere in three-dimensional space [5,6].

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