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Mathematical Analysis of Trigonometric Functionsa and Their Uses In Mathematics

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Abstract: Trigonometric functions, including sine, cosine, tangent, and their reciprocals, are fundamental components of mathematical analysis with extensive applications in various fields. This abstract delves into the mathematical properties, analytical methods, and practical uses of trigonometric functions. Initially defined as ratios of sides in right-angled triangles, these functions have been extended to the unit circle, enabling a broader scope of applications and a deeper understanding of their periodic nature and symmetries.
The analysis of trigonometric functions encompasses their behavior, transformations, and relationships. Key aspects include their periodicity, amplitude, phase shifts, and frequency, all of which are crucial in signal processing, Fourier analysis, and differential equations. Furthermore, the inverse trigonometric functions provide critical insights into solving equations involving trigonometric expressions. Applications of trigonometric functions are manifold in pure and applied mathematics. They are indispensable in geometry for solving triangles, in calculus for integration and differentiation, and in linear algebra for rotations and transformations. Beyond these, they serve pivotal roles in physics for wave motion, in engineering for signal processing and electrical circuits, and in computer science for graphics and simulations.

Keywords: Mathematical Analysis, Trigonometric Functions, Mathematics

Introduction

Trigonometric functions, such as sine, cosine, and tangent, are central to the field of mathematics due to their foundational role in describing relationships within triangles and periodic phenomena. Originating from the study of right-angled triangles, these functions have evolved to be described in terms of the unit circle, broadening their applicability to a wide array of mathematical and real-world problems.[1].

The sine and cosine functions define the coordinates of a point on the unit circle, providing a basis for understanding their periodic nature. Tangent, and the other trigonometric functions derived from sine and cosine, further enrich this framework. The periodicity, symmetry, and wave-like properties of these functions make them essential tools in both theoretical and applied mathematics. [2],

The study of trigonometric functions encompasses several key aspects:

1. **Behavior and Properties:** Understanding the periodicity, amplitude, and phase shifts.
2. **Analytical Techniques:** Applying differentiation and integration to solve problems involving trigonometric functions.
3. **Transformations and Inverses:** Exploring transformations such as scaling and translations, and solving equations using inverse trigonometric functions.

Trigonometric functions play a crucial role in numerous mathematical areas:

- **Geometry:** Solving problems related to angles and distances.
- **Calculus:** Facilitating the evaluation of integrals and derivatives involving periodic functions.
- **Linear Algebra:** Assisting in the representation of rotations and oscillations.

Beyond pure mathematics, trigonometric functions have extensive applications in various scientific and engineering fields:

- **Physics:** Modeling wave motion, oscillations, and harmonic analysis.
- **Engineering:** Signal processing, analyzing electrical circuits, and mechanical vibrations.
- **Computer Science:** Computer graphics, simulations, and animations.

The continuous development and exploration of trigonometric functions not only deepen our understanding of mathematical concepts but also drive innovations across multiple disciplines. This introduction sets the stage for a comprehensive analysis of the mathematical properties, techniques, and vast applications of trigonometric functions, illustrating their indispensable role in both theoretical and applied contexts.

Materials and Methods

Literature Review

he study of trigonometric functions has a rich historical and academic background, spanning several centuries and evolving through various mathematical advancements. This literature review synthesizes key developments and contemporary research in the analysis and applications of trigonometric functions.[3].

Historical Foundations

The origins of trigonometric functions can be traced back to ancient civilizations, with significant contributions from Greek mathematicians such as Hipparchus and Ptolemy, who developed early trigonometric tables and theories. The work of Indian mathematicians like Aryabhata and Bhaskara further advanced trigonometry, introducing concepts like sine and cosine. The Persian mathematician Al-Khwarizmi played a crucial role in transferring these ideas to the Islamic world, which later influenced European scholars during the Renaissance [4].

Classical Developments

In the 17th century, trigonometric functions were formalized through the work of mathematicians such as Isaac Newton and Gottfried Wilhelm Leibniz, who applied these functions in their development of calculus. The introduction of the unit circle by Euler provided a more comprehensive framework for understanding trigonometric functions, defining them as coordinates on a circle rather than merely ratios of sides in a triangle. Euler’s formula $eix=\cos(x)+i\sin(x)eix=\cos(x)+i\sin(x)$ bridged the gap between trigonometry and complex analysis, further enriching the field. [5]

Analytical Properties and Techniques

Modern mathematical analysis has deepened the understanding of trigonometric functions. Texts like “Trigonometric Delights” by Eli Maor and “A Course in Modern Mathematical Physics” by Peter Szekeres explore the intricate properties and applications of these functions. Research focuses on their periodicity, amplitude, and phase shifts, and their role in solving differential equations and evaluating integrals. Techniques such as Fourier analysis, as discussed in texts like "Fourier Analysis: An Introduction" by Elias Stein and Rami Shakarchi, highlight the importance of trigonometric functions in decomposing signals into their constituent frequencies.[4]

Applications in Various Fields

Trigonometric functions are indispensable in various scientific and engineering disciplines:

- **Physics:** Applications are detailed in works like "Classical Mechanics" by Herbert Goldstein, which uses trigonometric functions to model wave motion and oscillations.
- **Engineering:** "Signals and Systems" by Alan V. Oppenheim and Alan S. Willsky illustrates their use in signal processing and electrical circuits.
- **Computer Science:** Books like "Computer Graphics: Principles and Practice" by John F. Hughes et al. demonstrate the role of trigonometric functions in rendering graphics and animations.

Contemporary Research

Current research continues to expand the applications and theoretical understanding of trigonometric functions. Studies in fields such as quantum mechanics, digital signal processing, and machine learning explore novel uses of these functions. For instance, research articles in journals like the "Journal of Mathematical Analysis and Applications" and "IEEE Transactions on Signal Processing" frequently address advanced topics involving trigonometric functions, including their role in modern algorithms and computational methods.

Trigonometric Functions Topic Guide

The mathematics syllabuses are the documents used to inform the scope of content that will be assessed in the HSC examinations.

Topic Guides provide support for the Stage 6 Mathematics courses. They contain information organized under the following headings: Prior learning; Terminology; Use of technology; Background information; General comments; Future study; Considerations and teaching strategies; Suggested applications and exemplar questions. [6]

Topic Guides illustrate ways to explore syllabus-related content and consequently do not define the scope of problems or learning experiences that students may encounter through their study of a topic. The terminology list contains terms that may be used in the teaching and learning of the topic. The list is not exhaustive and is provided simply to aid discussion.

Please provide any feedback to the Mathematics and Numeracy Curriculum Inspector.

Results and Discussion

Topic focus

The topic Trigonometric Functions involves the study of periodic functions in geometric, algebraic, numerical and graphical representations.

A knowledge of trigonometric functions enables the solving of practical problems involving the manipulation of trigonometric expressions to model behaviour of naturally occurring periodic phenomena such as waves and signals and to predict future outcomes[7].

Study of trigonometric functions is important in developing students' understanding of periodic functions.

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Utilising the properties of periodic functions, mathematical models have been developed that describe the behaviour of many naturally occurring periodic phenomena, such as vibrations or waves, as well as oscillatory behaviour found in pendulums, electric currents and radio signals.

Terminology

amplitude	frequency	roots
angles of any measure	graph	related angle
angular measure	horizontal shift	sinusoidal functions
centre of motion	oscillation	sketch
circular measure	trigonometric identity	supplementary angles
constant	independent variable	symmetry properties
complementary angles	intercept	transformational shifts
composition	period	transformations
degrees	periodic	trigonometric function
dependent variable	phase shift	unit circle
derivative	quadrant	vertical shift
dilation	radian	wavelength
domain	range	wave form
exact ratio		

Use of technology

While 'by-hand' skills for solving equations and curve sketching are essential for students in this course, graphing technologies are an ideal means of exploring many of the concepts studied in this topic and their use is encouraged in teaching and learning.

In particular, graphing software is useful for investigating the effect of varying the constants a, b, c and k in the graph of $y = kf(a(x + b)) + c$ where $f(x)$ is a trigonometric function. [10]

Background information

The sine and cosine functions are called sinusoidal functions. They graph wave-forms and are used to describe any physical phenomenon that exhibits a wave-like pattern or periodic behaviour. Examples include the number of daylight hours at a specific location, the oscillation of a pendulum or the amount of energy used to control the temperature in an office.

An early application of sinusoidal functions was to predict the tides, providing important information to those involved in coastal navigation and the fishing industry. The link between the tides and the gravitational pull of the Sun and Moon on the oceans had been known for many centuries. The suggestion that they may also be periodic prompted the use Fourier analysis to build tide-predicting machines.

Sound waves are created through vibrations that consist of wavelength, frequency, velocity and amplitude, and consequently they can also be modelled by sinusoidal functions. Sound waves are characterised as mechanical waves because they are a disturbance that is transported through a medium. They cannot travel through a vacuum.[8]

As the heart beats and blood is pumped through the body, blood flows through the arteries in a pattern similar to a sinusoidal function.

General comments

The material in this topic builds on the content from the Measurement and Geometry strand of the *K–10 Mathematics* syllabus and related content from the *Mathematics Advanced* syllabus, including the Year 11 topics of Functions and Trigonometric Functions.
This topic prepares students for many practical applications of trigonometric functions and is essential for many of the more advanced aspects of mathematics.
As with the study of Graphing Techniques in MA-F2, importance must be placed on the order in which the transformations are applied to the original function, and their effect on the position and shape of the graph. This could be investigated dynamically using graphing software.
Both sine and cosine graphs are referred to as sinusoidal graphs, because $\cos x = \sin(\frac{\pi}{2} - x)$, and so the graph of $f(x) = \cos x$ is effectively a sine wave with a phase shift of $\frac{\pi}{2}$ radians.

Future study

This topic could be taught in conjunction with the Functions topic MA-F2 Graphing Techniques where the transformational shifts of graphs are taught in the context of other functions.

MA-T3: Trigonometric Functions and Graphs

Subtopic focus

The principal focus of this subtopic is to explore the key features of the graphs of trigonometric functions and to understand and use basic transformations to solve trigonometric equations.
Students develop an understanding of the way that graphs of trigonometric functions change when the functions are altered in a systematic way. This is important in understanding how mathematical models of real-world phenomena can be developed[9] .

Considerations and teaching strategies

Review of the following may be needed to meet the needs of students:

- Angle measures, representations and conversions – This relates to content covered in MA-T1 (T1.2).
- The graphs of $y = \sin x$, $y = \cos x$ and $y = \tan x$ – This relates to content covered in MA-T1 (T1.2).

The original graphs of $y = \sin x$ and $y = \cos x$ are easily developed from the definitions based on the unit circle and could be illustrated using appropriate graphing software applications as a brief review.

Both radian measure and degrees should be used within this topic.

The domain and range, period and amplitude of simple trigonometric functions should be noted from graphs plotted using computer software, or 'by-hand' methods.

Initially, examples of sketching related curves of the form $y = kf(a(x + b)) + c$ should be restricted to changing the value of one variable at a time, for example:

(a) Sketch the functions $f_1(x) = \sin x$ and $f_2(x) = \sin 3x$ on the same axes.

(b) Sketch the functions $g(x) = \cos x$ and $h(x) = 2 + \cos x$ on the same axes.

Once these concepts are understood this can be extended to the consideration of more than one transformation at a time [10].

The order in which the transformations are applied should be considered when transformations are combined. This can be effectively investigated using graphing software[8].

Equivalent expressions for sinusoidal functions include $y = \sin\left(\frac{2\pi x}{P}\right)$, where P is the period or wavelength, and $y = \sin(2\pi\omega x)$, where ω is the frequency.

Sketches of functions such as $y = 3\cos(2x)$, $y = \sin \pi x$ and $y = 1 - \cos\frac{x}{2}$ should be drawn by hand, showing the main features.

Students should be given some practice in using graphs to solve simple equations such as $\sin 2x = \frac{1}{2}x$.

Real-world applications should be discussed, for example data on daily temperatures and other periodic phenomena can be plotted and amplitudes and periods estimated from the graphs.

It should be noted that real-world tides are a superposition of a number of different sinusoidal functions, with different amplitudes and frequencies, which makes real-world tidal data difficult to model by a single sinusoidal function. However, a useful discussion can be had around this point.
More information on tidal constituents can be found on the US National Oceanic and Atmospheric Administration (noaa) website at: https://tidesandcurrents.noaa.gov/harcon.html?id=9410170.

Modelling of weather patterns and other geographical data often involves sinusoidal functions. For example, the number of daylight hours on the t^{th} day of the year for a particular city can be modelled by a function of the form $L(t) = 12 + a\sin\left[\frac{2\pi}{365}(t - b)\right]$, where a and b are constants.

For sound waves, the loudness of a sound depends on the amplitude of the wave and the pitch of a sound depends on the frequency of the wave.

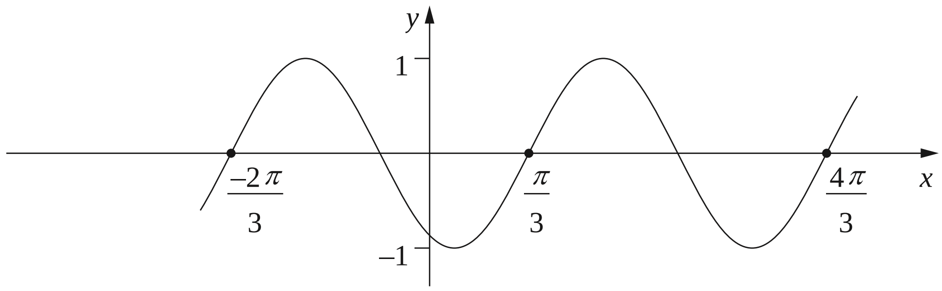
Real-world applications of adding sinusoidal graphs together can be found in biorhythms, harmonics, cardiographs, tides, etc.

Given a graph showing changes in a real-world phenomenon and its equation, students could describe the oscillation in words[5].

Suggested applications and exemplar questions

Which diagram shows the graph of $y = \sin(2x + \frac{\pi}{3})$?

(A)



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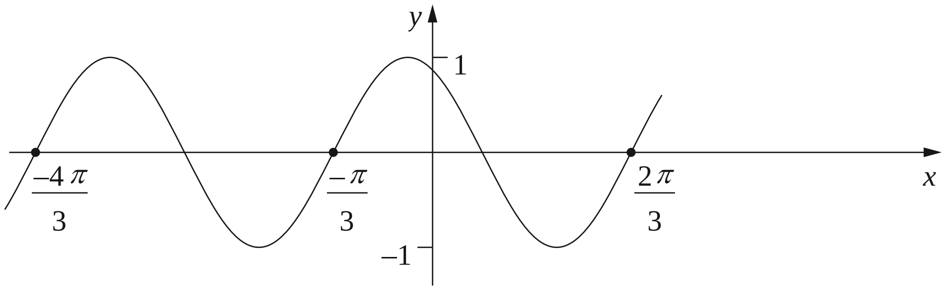
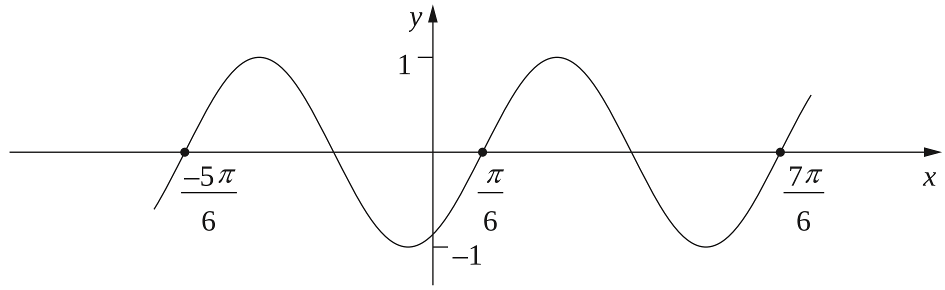
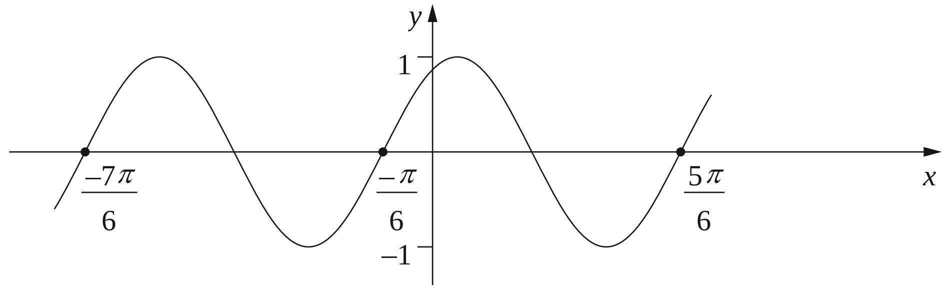
(B)



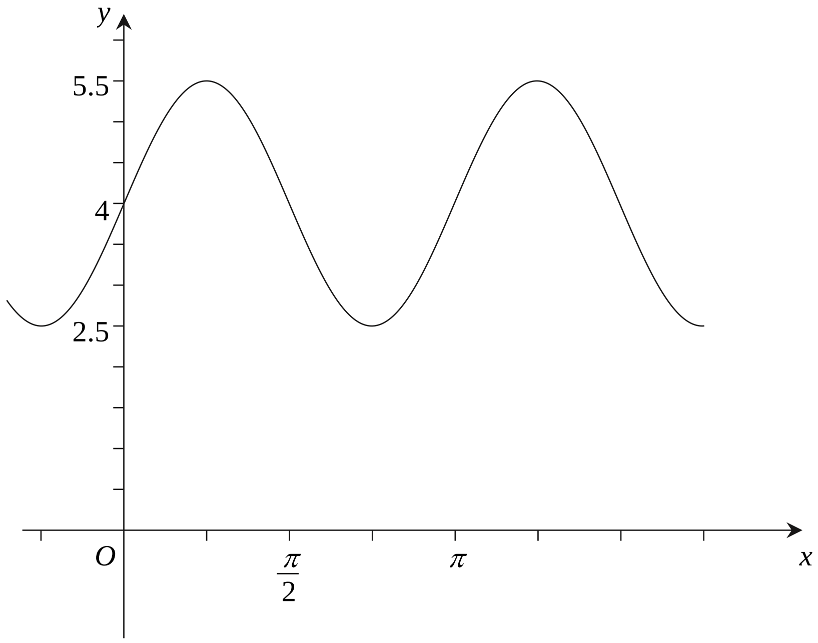
(C)



(D)



The diagram shows part of the graph of $y = a\sin(bx) + 4$.



What are the values of a and b?

A. $a = 3\ b = \frac{1}{2}$

B. $a = 3\ b = 2$

C. $a = 1.5$ $b = \frac{1}{2}$

D. $a = 1.5$ $b = 2$

What is the period of the function $f(x) = \tan(3x)$?

A. $\frac{\pi}{3}$

B. $\frac{2\pi}{3}$

C. 3π

D. 6π

Sketch the curve $y = 1 - \sin 2x$ for $0 \leq x \leq \pi$.

(a) Sketch the graph of $y = 2\cos x$ for $0 \leq x \leq 2\pi$.

(b) On the same set of axes, sketch the graph of $y = 2\cos x - 1$ for $0 \leq x \leq 2\pi$.

(c) Find the exact values of the x coordinates of the points where the graph of $y = 2\cos x - 1$ crosses the x-axis in the domain $0 \leq x \leq 2\pi$.

Solve $\sin\left(\frac{x}{2}\right) = \frac{1}{2}$ for $0 \leq x \leq 2\pi$.

Solve $2\sin^2 x - 3\sin x - 2 = 0$ for $0 \leq x \leq 2\pi$.

(a) Draw the graphs of $y = 4\cos x$ and $y = 2 - x$ on the same set of axes for $-2\pi \leq x \leq 2\pi$.

(b) Explain why all the solutions of the equation $4\cos x = 2 - x$ must lie between $x = -2$ and $x = 6$.

(a) Show that $x = \frac{\pi}{3}$ is a solution of $\sin x = \frac{1}{2}\tan x$.

(b) On the same set of axes, sketch the graphs of the functions $y = \sin x$ and $y = \frac{1}{2}\tan x$ for $-\pi \leq x \leq \pi$.

(c) Hence find all solutions of $\sin x = \frac{1}{2}\tan x$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

(d) Use your graphs to solve $\sin x \leq \frac{1}{2}\tan x$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

The graph of $y = 3\cos(2x + \alpha)$ can be obtained from the graph of $y = \cos x$ by a translation followed by two dilations.

(a) Describe each of these three transformations, and give the number of roots of the equation $3\cos(2x + \alpha) = k$, in the interval $0 \leq x < 2\pi$, where $-3 < k < 3$.

(b) Generalise your answer to give the number of roots of the equation $a\cos(nx + \alpha) = k$ in the interval $0 \leq x < 2\pi$, where $a < 0$, n is a positive integer and $-a < k < a$.

(c) How does your answer to part (b) change if n is a negative integer?

A particle moves in a straight line. At time t seconds its distance x metres from a fixed point O in the line is given by $x = 2 - 2\cos 2t$.

(a) Sketch the graph of x as a function of t.

(b) Find the times when the particle is at rest and the position of the particle at those times.

(c) Describe the motion.

The length of daylight, $L(t)$, is defined as the number of hours from sunrise to sunset, and can be modelled by the equation $L(t) = 12 + \cos\left(\frac{2\pi t}{366}\right)$ where t is the number of days after 21 December 2015, for $0 \leq t \leq 366$.

(a) Find the length of daylight on 21 December 2015.

(b) What is the shortest length of daylight?

(c) What are the two values of t for which the length of daylight is 11?

When humans breathe, they do not inflate their lungs to full capacity. When resting, each inhalation adds approximately 0.5 L of air and this same volume of air is removed upon exhalation. When exhalation is completed, the volume of air that remains in the lungs, called the functional residual capacity, is approximately 2.2 L. On average the time taken to complete an inhale-exhale cycle is approximately 5 seconds.

The volume of air in the lungs can be modelled by the function $V = k\sin(at) + c$ where V is the volume of air in litres and t is time in seconds.

(a) Use the time for an inhale-exhale cycle to show that the period of this function is $\frac{2\pi}{5}$.

(b) Explain why $k = 0.25$.

(c) Find the value of c.

(d) Sketch the graph of $V = k\sin(at) + c$ for $0 \leq t \leq 15$ using these values of k, a and c.

(e) When exercising, the volume of air inhaled and exhaled rises and breathing occurs more rapidly. Explain the effect this would have on the values of k, a and c.

(f) Humans have a full lung capacity of approximately 6 L. An athlete who is exercising vigorously inhales approximately 4.6 L of air. Calculate the athlete's residual lung capacity.

Conclusion

In conclusion, the mathematical analysis of trigonometric functions reveals their fundamental importance and wide-ranging applications within various fields of mathematics and beyond. Trigonometric functions, including sine, cosine, and tangent, serve as essential tools in numerous mathematical disciplines. Their properties and relationships, such as periodicity, symmetry, and orthogonality, are crucial for solving problems in both pure and applied mathematics.[11].

Key Points

1. **Foundation of Periodic Phenomena**: Trigonometric functions are inherently periodic, making them invaluable for modeling periodic phenomena in nature and engineering. They are used extensively in Fourier analysis, which decomposes complex periodic functions into sums of sines and cosines, aiding in signal processing and the study of waveforms.

2. **Geometric Interpretations**: The geometric interpretations of trigonometric functions in the context of the unit circle provide insights into the properties of angles and distances. This geometric perspective is fundamental in fields such as calculus, where these functions help in the differentiation and integration of periodic functions.

3. **Complex Numbers and Euler's Formula**: Trigonometric functions bridge real and complex analysis, particularly through Euler's formula $eix=\cos(x)+i\sin(x)eix=\cos(x)+i\sin(x)$. This connection is pivotal in complex analysis, offering a powerful framework for understanding complex exponential functions and their applications.

4. **Applications in Differential Equations**: In the study of differential equations, trigonometric functions often arise as solutions to linear differential equations with constant coefficients. They model phenomena such as harmonic oscillators, electrical circuits, and mechanical vibrations.

5. **Coordinate Transformations**: Trigonometric functions facilitate coordinate transformations, particularly in converting between Cartesian and polar coordinates. This is essential in multivariable calculus and analytical geometry for simplifying the analysis of curves and surfaces.

6. **Optimization and Approximations**: The critical points of trigonometric functions are used in optimization problems and numerical methods. They also play a role in approximations, such as Taylor and Maclaurin series, where trigonometric series can approximate more complex functions.

Broader Implications

1. Beyond pure mathematics, trigonometric functions have extensive applications in physics, engineering, computer graphics, and other sciences. They model sound waves, light waves, and electromagnetic waves, describe the motion of pendulums, and are integral in algorithms for computer graphics and visualizations. Their ubiquitous presence underscores the profound impact of trigonometric analysis on both theoretical and practical advancements.

2. In summary, the study of trigonometric functions is a cornerstone of mathematical education and research. Their analysis not only deepens our understanding of mathematical concepts but also enhances our ability to model and solve real-world problems across various scientific and engineering domains.

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