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Exponential Reliability Estimation of (N+1) Cascade Model

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Abstract

In this paper presents the R reliability mathematical formula of (N+1) Exponential Cascade model . The reliability of the model is expressed by Exponential random variables,which are stress and strength distributions. The reliability model was estimated by seven dissimilar methods (ML , Mo , LS , WLS , Rg ,Pr and Pi) and simulation was performed using MATLAB 2012 program to compare the results of the reliability model estimates using the MSE criterion , the results indicated that the best estimator among the six estimators was ML , MO and Pi.

Keywords: Standby redundancy, Parameter, Exponential distribution, Unit, distributed identically.

1. Introduction:

Many researches have been performed on reliability estimation $R = p(X > Y)$ in the field of strength and stress models . The Cascade is a special kind of stress-strength model . Cascade redundancy is a hierarchical standby redundancy in which a standby unit with different stress substitutes for a system . When a system unit fails , it is replaced by a standby unit and the stress changed k Times the previous stress [10] .

In a previous study Karam and Khaleel (2019) presented a study of (2+1) Cascade model, which the model consists of two main components and one redundancy standby . In this paper , we assumed that the (3+1) of Cascade with $(U_1 , U_2 , U_3 \ and \ U_4)$ Units ,in which three units $U_1 , U_2 , U_3 \ and \ U_4$ are work and the unit U_4 is a standby unit. Assume that X_1 , X_2 , X_3 , X_4 denote the unit strengths $(U_1 , U_2 , U_3 \ and \ U_4)$respectively and Y_1 , Y_2 , Y_3 , Y_4 indicated the enforcement of stress . Here , if the active unit U_1 is a failure then the standby component U_4 is activated , where $X_4 = \mathrm{m}X_1 \ and \ Y_4 = kY_1$, if the active unit U_1 is a failure then the standby component U_4 is activated , where $X_4 = \mathrm{m}X_2 \ and \ Y_4 = kY_2$ and if the active unit R_3 is a failure then the standby component R_4 is activated , where $X_4 = \mathrm{m}X_3 \ and \ Y_4 = kY_3$ Where " k " and " m " denote the stress and strength attenuation factors respectively , such that $0 < m < 1$ and $k > 1$ Reddy (2016) [18] presents of $R = p(X > Y)$ by discussing model stress – strength of a cascade , assuming all the parameters are independent and following Weibull stress-strength distribution in one parameter and calculating first four cascade reliability for different stress-strength values. Mutkekar and Munoli (2016)[15] , (1+1) exponential distribution cascade model is derived with the common effect of the force and stress reduction factors . Kumar and Vaish (2017) [13] , discussed that Gompertz distribution is stress and that strength is power distribution parameters . Karam and Khaleel (2018) [10] derived a special (2+1) stress-strength reliability cascade model for the distribution of Weibull . Khaleel and Karam (2019) [12] discussed the reliability of the (2+1) cascade inverse distribution Weibull model , reliability can be found when reverse Weibull random variables with unknown parameters scale and known shape parameter are distributed with strength-stress and used six different estimations mothed to estimate reliability . Karam and Khaleel (2019) [9] , expression for model confidence is found when strength and stress distribution are generalized in reversed Rayleigh random variable Rayleigh , derived from mathematical formulas for Reliability to Special (2+1) . Khaleel (2021)[11] , (3+1) exponential distribution cascade model is derived with the common effect of the force and stress reduction factors.

2. The mathematical formula:

Suppose, for the four units (three basic and one redundant standby), the random strength-stress variables of the four units $j = 1,2,3,4$ each independently and identically distributed Exponential of the parameter scale β_i , i =1,2,3,4 and scale μ_j , j =1,2,3,4

2.1 . Exponential Distribution:[11]

The exponential distribution is one of the widely used continuous distributions. It is often used to model the time elapsed between events.

The PDF of $Ex(\beta)$ **:**

$$f(x,\sigma) = \frac{1}{\sigma} e^{\frac{-x}{\sigma}} \quad ,\sigma > 0 \,, x > 0$$

Let $\beta = \frac{1}{\sigma}$

$$\rightarrow \; f(x,\beta) = \beta e^{-\beta x} \qquad ,x > 0 \;\;, \;\; \beta > 0 \qquad \dots(1)$$

The reliability function of $Ex(\beta)$

$$R(x) = e^{-\beta x} \qquad \dots(2)$$

hazard function of $EP(\theta,\alpha,\beta)$:

$$h(x) = \beta \qquad \dots(3)$$

The Mean of $Ex(\beta)$:

$$E(x) = \frac{1}{\beta} \qquad \dots(4)$$

The Variance of $Ex(\beta)$:

$$V(x) = \frac{1}{\beta^2} \qquad \dots(5)$$

The Cumulative distribution function of $Ex(\beta)$ is :

$$F(x) = 1 - e^{-\beta x} \quad ,x > 0 \,,\beta > 0 \qquad \dots(6)$$

The Cumulative distribution function of $Ex(\mu)$ is :

$$G(y) = 1 - e^{-\mu y} \qquad ,y > 0 \,,\mu > 0 \qquad \dots(7)$$

2.2. Reliability Model for Exponential Distribution (R_{Ex}):

Let $X_i \sim Ex(\beta_i); i = 1,2,3,4$ and $Y_j \sim Ex(\mu_j); j = 1,2,3,4$ be strength and stress random variables of the three components (three components are basic and one is standby)with unknown scale parameters β_i , μ_j , where X_i and Y_j are independently and identically distributed Exponential random variables .

The reliability function for (N+1) cascade model is :

$$R = P[X_1 \geq Y_1, X_2 \geq Y_2, X_3 \geq Y_3, \dots, X_N \geq y_N]$$
$$+P[X_1 < Y_1, X_2 \geq Y_2, X_3 \geq Y_3, \dots, X_{N+1} \geq Y_{N+1}]$$
$$+P[X_1 \geq Y_1, X_2 < Y_2, X_3 \geq Y_3, \dots, X_{N+1} \geq Y_{N+1}]$$
$$+P[X_1 \geq Y_1, X_2 \geq Y_2, X_3 < Y_3, X_4 \geq Y_4, \dots, X_{N+1} \geq Y_{N+1}]$$
$$+ \dots + P[X_1 \geq Y_1, X_2 \geq Y_2, X_3 \geq Y_3, \dots, X_N < Y_N, X_{N+1} \geq Y_{N+1}]$$
$$R = R_1 + R_2 + R_3 + \dots + R_{N+1} \quad \dots (8)$$

$$P_r = P(X_r \geq Y_r) = \int_{y_r} [1 - F_r(y_r)] f(y_r)\, dy_r \quad , r \in \{1,2,3,\dots,N\}$$
$$= \int_0^\infty \left(e^{-\beta_r y_r}\right)\left(\mu_r e^{-\mu_r y_r}\right) dy_r$$
$$= \int_0^\infty \mu_r e^{-(\beta_r+\mu_r)y_r}\, dy_r = \int_0^\infty \frac{-(\beta_r+\mu_r)}{-(\beta_r+\mu_r)} \mu_r e^{-(\beta_r+\mu_r)y_r}\, dy_r$$
$$= \frac{-\mu_r}{\beta_r+\mu_r} \int_0^\infty -(\beta_r+\mu_r) e^{-(\beta_r+\mu_r)y_r}\, dy_r$$
$$= \frac{-\mu_r}{\beta_r+\mu_r} \left[e^{-(\beta_r+\mu_r)y_r}\right]_0^\infty = (0) - \left(\frac{-\mu_r}{\beta_r+\mu_r}\right) = \frac{\mu_r}{\beta_r+\mu_r} \quad \dots (9)$$

$$P_{-r} = P[X_r < Y_r, X_{N+1} \geq Y_{N+1}]$$
$$= \int_{y_r} F_r(y_r)\left[1 - F_r\left(\frac{k}{m} y_r\right)\right] f(y_r) dy_r \quad , r \in \{1,2,3,\dots,N\}$$
$$= \int_0^\infty \left(1 - e^{-\beta_r y_r}\right)\left(e^{-\beta_r \frac{k}{m} y_r}\right)\left(\mu_r e^{-\mu_r y_r}\right) dy_r$$
$$= \int_0^\infty \left(e^{-\beta_r \frac{k}{m} y_r}\right)\left(\mu_r e^{-\mu_r y_r}\right) - \left(e^{-\beta_r y_r}\right)\left(e^{-\beta_r \frac{k}{m} y_r}\right)\left(\mu_r e^{-\mu_r y_r}\right) dy_r$$
$$= \int_0^\infty \mu_r e^{-\left(\mu_r + \frac{k}{m}\beta_r\right) y_r} - \mu_r e^{-\left[\left(1+\frac{k}{m}\right)\beta_r + \mu_r\right] y_r}\, dy_r$$
$$= \int_0^\infty \mu_r \frac{-\left(\mu_r + \frac{k}{m}\beta_r\right)}{-\left(\mu_r + \frac{k}{m}\beta_r\right)} e^{-\left(\mu_r + \frac{k}{m}\beta_r\right) y_r} - \frac{\left(1+\frac{k}{m}\right)\beta_r + \mu_r}{\left(1+\frac{k}{m}\right)\beta_r + \mu_r} \mu_r e^{-\left[\left(1+\frac{k}{m}\right)\beta_r + \mu_r\right] y_r}\, dy_r$$
$$= \left[\frac{-\mu_r}{\mu_r + \frac{k}{m}\beta_r} e^{-\left(\mu_r + \frac{k}{m}\beta_r\right) y_r} + \frac{\mu_r}{\left(1+\frac{k}{m}\right)\beta_r + \mu_r} e^{-\left[\left(1+\frac{k}{m}\right)\beta_r + \mu_r\right] y_r}\right]_0^\infty$$

$$= (0+0) - \left(\frac{-\mu_r}{\mu_r + \frac{k}{m}\beta_r} + \frac{\mu_r}{\left(1+\frac{k}{m}\right)\beta_r + \mu_r} \right)$$

$$= \frac{\mu_r}{\mu_r + \frac{k}{m}\beta_r} - \frac{\mu_r}{\left(1+\frac{k}{m}\right)\beta_r + \mu_r}$$

$$= \frac{\beta_r \mu_r}{\left(\mu_r + \frac{k}{m}\beta_r\right)\left[\left(1+\frac{k}{m}\right)\beta_r + \mu_r\right]} \quad \dots (10)$$

2.3 Remark:

i)

$$R_g = P_{1:g} P_{2:g} \dots P_{N:g} \qquad , g \in \{1,2,\dots,N+1\} \quad \dots (11)$$

$$P_{r:g} = \int_{y_r} [F_r(y_r)]^h \left[1 - F_r\left(\left(\frac{k}{m}\right)^h y_r\right)\right] f(y_r) dy_r \qquad , r \in \{1,2,3,\dots,N\}$$

$$= \begin{cases} P_r & , h = 0 \\ P_{-r} & , h = 1 \end{cases} = \begin{cases} P_r & , t > 0 \\ P_{-r} & , t = 0 \end{cases} \quad \dots (12)$$

Such that

$$t = |r - g + 1| \qquad , g \in \{1,2,\dots,N+1\} \quad \dots (13)$$

$$h = \frac{1 - t + |1 - t|}{2} = \begin{cases} 0 , & t > 0 \\ 1 , & t = 0 \end{cases} \quad \dots (14)$$

$$R_1 = P_1 P_2 P_3 \dots P_N \quad \dots (15)$$

$$R_2 = P_{-1} P_2 P_3 \dots P_N = \frac{P_{-1}}{P_1} R_1 \quad \dots (16)$$

$$R_3 = P_1 P_{-2} P_3 \dots P_N = \frac{P_{-2}}{P_2} R_1 \quad \dots (17)$$

$$R_4 = P_1 P_2 P_{-3} \dots P_N = \frac{P_{-3}}{P_3} R_1 \quad \dots (18)$$

$$\vdots$$

$$R_{N+1} = P_1 P_2 P_3 \dots P_{-N} = \frac{P_{-N}}{P_N} R_1 \quad \dots (19)$$

$$R = R_1 + R_2 + R_3 + R_4 + \dots + R_{N+1}$$

$$= R_1 + \frac{P_{-1}}{P_1} R_1 + \frac{P_{-2}}{P_2} R_1 + \frac{P_{-3}}{P_3} R_1 + \dots + \frac{P_{-N}}{P_N} R_1$$

$$= R_1 \left(1 + \frac{P_{-1}}{P_1} + \frac{P_{-2}}{P_2} + \frac{P_{-3}}{P_3} + \dots + \frac{P_{-N}}{P_N}\right)$$

$$= P_1 P_2 P_3 \dots P_N \left(1 + \frac{P_{-1}}{P_1} + \frac{P_{-2}}{P_2} + \frac{P_{-3}}{P_3} + \dots + \frac{P_{-N}}{P_N}\right)$$

$$= \left(\prod_{r=1}^{N} P_r\right)\left(1 + \sum_{r=1}^{N} \frac{P_{-r}}{P_r}\right)$$

$$\rightarrow R = \left(\prod_{r=1}^{N} P_r\right)\left(1 + \sum_{r=1}^{N} \frac{P_{-r}}{P_r}\right)$$

$$= \left(\prod_{r=1}^{N} \frac{\mu_r}{\beta_r + \mu_r}\right)\left(1 + \sum_{r=1}^{N} \frac{\left[\frac{\beta_r \mu_r}{\left(\mu_r + \frac{k}{m}\beta_r\right)\left[\left(1 + \frac{k}{m}\right)\beta_r + \mu_r\right]}\right]}{\left(\frac{\mu_r}{\beta_r + \mu_r}\right)}\right)$$

$$= \left(\prod_{r=1}^{N} \frac{\mu_r}{\beta_r + \mu_r}\right)\left(1 + \sum_{r=1}^{N} \frac{\beta_r(\mu_r + \beta_r)}{\left(\mu_r + \frac{k}{m}\beta_r\right)\left[\left(1 + \frac{k}{m}\right)\beta_r + \mu_r\right]}\right) \quad \dots (20)$$

ii)

$$R = \left(\prod_{r=1}^{N} \frac{\mu_r}{\beta_r + \mu_r}\right)\left(1 + \sum_{r=1}^{N} \frac{\beta_r(\mu_r + \beta_r)}{\left(\mu_r + \frac{k}{m}\beta_r\right)\left[\left(1 + \frac{k}{m}\right)\beta_r + \mu_r\right]}\right)$$

Let $\beta_i = \beta_j = \beta \quad and \quad \mu_i = \mu_j = \mu \quad ,\forall \ i,j \in \{1,2,3,\dots,N\}$

$$\rightarrow R = \left(\frac{\mu}{\beta + \mu}\right)^N \left(1 + \frac{N\beta(\mu + \beta)}{\left(\mu + \frac{k}{m}\beta\right)\left[\left(1 + \frac{k}{m}\right)\beta + \mu\right]}\right) \quad \dots (21)$$

3.Parameters Estimation of Weibull distribution.

3.1 Maximum Likelihood Estimation Method (ML):

Making the maximum likelihood was one of most important developments in 20th century statistics . In (1922) Fisher introduced the method of maximum likelihood . He first presented the numerical procedure in (1912) , but in (1922) the maximum likelihood method gave estimates satisfying the criteria of efficiency and sufficiency and there were two forms for sometimes Fisher based the likelihood on the distribution of the entire sample , sometimes on the distribution of a specific statistic .[3]

Suppose that a random sample $X_1, X_2, X_3, \dots, X_n$ have $Ex(\beta)$ distribution with sample size n , where β is unknown scale parameter, then the likelihood function "L", the joint probability function with the general form , can be written as follows :[5]

$$L(X_1, X_2, \dots, X_n, \beta) = f(X_1, \beta) f(X_2, \beta) \dots f(X_n, \beta) = \prod_{i=1}^{n} f(X_i, \beta)$$

Then likelihood function using equation (1) will be as :

$$L(X_1, X_2, \dots, X_n, \beta) = \prod_{i=1}^{n} [\beta e^{-\beta x_i}]$$

$$L(X_1, X_2, \dots, X_n, \beta) = \beta^n e^{-\beta \sum_{i=1}^{n} x_i} \qquad \dots(22)$$

Then natural logarithm function for equation (22) can be written as ;

$$lnL = ln\left[\beta^n e^{-\beta \sum_{i=1}^{n} x_i}\right]$$

$$lnL = nln\beta - \beta \sum_{i=1}^{n} x_i \qquad \dots(23)$$

To minimize , natural logarithm in equation (23) , must compute the great endings by taking partial derivative with respect to unknown scale parameter β, then will get as:

$$\frac{\partial lnL}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^{n} x_i \qquad \dots(24)$$

Equating partial derivative to zero , thus the right-hand side of (24) will be :

$$\rightarrow \frac{n}{\hat{\beta}} - \sum_{i=1}^{n} x_i = 0 \qquad \dots(25)$$

The maximum likelihood estimator for β is given by :

$$\rightarrow \hat{\beta}_{(ML)} = \frac{n}{\sum_{i=1}^{n} x_i} \qquad \dots(26)$$

In the same way above , let $Y_1, Y_2, Y_3, \dots, Y_m$ a random sample have $Ex(\mu)$ distribution with the sample size m , then the maximum likelihood estimator of unknown scale parameter μ ; says $\hat{\mu}_{(ML)}$; is :

$$\hat{\mu}_{(ML)} = \frac{m}{\sum_{j=1}^{m} y_j} \qquad \dots(27)$$

Now , suppose that $X_1 \sim Ex(\beta_1), X_2 \sim Ex(\beta_2)$, $X_3 \sim Ex(\beta_3), \dots, X_N \sim Ex(\beta_N)$ are strengths r.v.'s with the samples sizes n_1, n_2 , n_3 … n_N respectively , where $(\beta_1, \beta_2, \beta_3, \dots, \beta_N)$ are the unknown scale parameters and suppose that $Y_1 \sim W(\mu_1)$, $Y_2 \sim W(\mu_2)$, $Y_3 \sim W(\mu_3)$ … $Y_N(\mu_N)$ are the stresses r.v.'s with samples sizes $m_1, m_2, m_3,$ … m_N respectively , where $(\mu_1, \mu_2, \mu_3, \dots, \mu_N)$ are unknown scale parameters. By using the same way , the maximum likelihood estimators $(\beta_1, \beta_2, \beta_3, \dots, \beta_N)$ and $(\mu_1, \mu_2, \mu_3, \dots, \mu_N)$ are:

$$\hat{\beta}_{\delta(ML)} = \frac{n_\delta}{\sum_{i_\delta=1}^{n_\delta} X_{\delta_{i_\delta}}{}^{\alpha}} \quad , \delta = 1,2,3, \dots, N \qquad \dots(28)$$

and

$$\hat{\mu}_{\delta(ML)} = \frac{m_\delta}{\sum_{j_\delta=1}^{m_\delta} Y_{\delta_{j_\delta}}{}^{\alpha}} \quad , \delta = 1,2,3, \dots, N \qquad \dots(29)$$

Substituting (28) and (29) in (21) , the maximum likelihood estimator for reliability R ; $\hat{R}_{(ML)}$; invariability will be as :

$$\hat{R}_{Ex(ML)} = \hat{R}_{1(ML)} + \hat{R}_{2(ML)} + \hat{R}_{3(ML)} + \cdots + \hat{R}_{(N+1)(ML)}$$

$$= \left(\prod_{r=1}^{N} \frac{\hat{\mu}_{r(ML)}}{\hat{\beta}_{r(ML)} + \hat{\mu}_{r(ML)}}\right)\left(1 + \sum_{r=1}^{N} \frac{\hat{\beta}_{r(ML)}\left(\hat{\mu}_{r(ML)} + \hat{\beta}_{r(ML)}\right)}{\left(\hat{\mu}_{r(ML)} + \frac{k}{m}\hat{\beta}_{r(ML)}\right)\left[\left(1 + \frac{k}{m}\right)\hat{\beta}_{r(ML)} + \hat{\mu}_{r(ML)}\right]}\right)$$

$$= \left(\frac{\hat{\mu}_{(ML)}}{\hat{\beta}_{(ML)} + \hat{\mu}_{(ML)}}\right)^{N}\left(1 + \frac{N\hat{\beta}_{(ML)}\left(\hat{\mu}_{(ML)} + \hat{\beta}_{(ML)}\right)}{\left(\hat{\mu}_{(ML)} + \frac{k}{m}\hat{\beta}_{(ML)}\right)\left[\left(1 + \frac{k}{m}\right)\hat{\beta}_{(ML)} + \hat{\mu}_{(ML)}\right]}\right) \quad \ldots (30)$$

3.2 Moments Estimation Method (Mo):

Karl Pearson in (1894) introduced a formal approach to the statistical estimation through his method of moments (Mo) estimation . He quite unceremoniously suggested a method that simply equal the first five sample moments to the respective population counterparts . It was not simple to solve five highly the nonlinear equations . Therefore , he took an analytical approach of removing one parameter in all step . After considerable algebra Pearson found a ninth degree polynomial equation in unknown one . Then after solving the equation and by reiterated back substitutions , Pearson found the solutions to five parameters in the terms of the five first sample moments , and was beginning of the moments method (Mo) estimation [4] . To derive method of the moments estimator parameters of ED , assume that x_i , $i = 1,2,3,\ldots,n$ random sample have $Ex(\beta)$ distribution with the sample size n , first step the mean population of $Ex(\beta)$, obtain by equation (4):[8]

$$E(X) = \frac{1}{\beta}$$

The second step equating mean sample with corresponding the mean population , then will get as :

$$\frac{\sum_{i=1}^{n} X_i}{n} = \frac{1}{\beta} \quad \ldots (31)$$

Then the moment estimator of β says $\hat{\beta}_{(Mo)}$ is :

$$\hat{\beta}_{(MO)} = \frac{1}{\bar{X}} \quad \ldots (32)$$

In the same manner , the moments estimator of unknown scale parameter μ ; says $\hat{\mu}_{(Mo)}$; is :

$$\hat{\mu}_{(MO)} = \frac{1}{\bar{Y}} \quad \ldots (33)$$

Now , by using the same technique , the moments estimators of the unknown scale parameters ($\beta_1, \beta_2, \beta_3, \ldots, \beta_N$) and ($\mu_1, \mu_2, \mu_3, \ldots, \mu_N$) are:

$$\hat{\beta}_{\delta(MO)} = \frac{1}{\bar{x}_\delta} \quad , \ \delta = 1,2,3,\ldots,N \quad \ldots (34)$$

And

$$\hat{\mu}_{\delta(MO)} = \frac{1}{\bar{Y}_\delta} \quad , \quad \delta = 1,2,3,\ldots,N \quad \ldots (35)$$

Substitution (34) and (35) in (21) , the moments estimator for reliability R_{Ex} ; says $\hat{R}_{Ex(Mo)}$; approximately will be as:

$$\hat{R}_{Ex(Mo)} = \hat{R}_{1(Mo)} + \hat{R}_{2(Mo)} + \hat{R}_{3(Mo)} + \cdots + \hat{R}_{(N+1)(Mo)}$$

$$= \left(\prod_{r=1}^{N} \frac{\hat{\mu}_{r(Mo)}}{\hat{\beta}_{r(Mo)} + \hat{\mu}_{r(Mo)}}\right)\left(1 + \sum_{r=1}^{N} \frac{\hat{\beta}_{r(Mo)}\left(\hat{\mu}_{r(Mo)} + \hat{\beta}_{r(Mo)}\right)}{\left(\hat{\mu}_{r(Mo)} + \frac{k}{m}\hat{\beta}_{r(Mo)}\right)\left[\left(1 + \frac{k}{m}\right)\hat{\beta}_{r(Mo)} + \hat{\mu}_{r(Mo)}\right]}\right)$$

$$= \left(\frac{\hat{\mu}_{(Mo)}}{\hat{\beta}_{(Mo)} + \hat{\mu}_{(Mo)}}\right)^{N}\left(1 + \frac{N\hat{\beta}_{(Mo)}\left(\hat{\mu}_{(Mo)} + \hat{\beta}_{(Mo)}\right)}{\left(\hat{\mu}_{(Mo)} + \frac{k}{m}\hat{\beta}_{(Mo)}\right)\left[\left(1 + \frac{k}{m}\right)\hat{\beta}_{(Mo)} + \hat{\mu}_{(Mo)}\right]}\right) \quad \ldots (36)$$

3.3 Least Squares Estimation Method (LS):

The German mathematician Carl Friedrich Gauss had inspected the least squares as early in (1794) , he did not publish the method until (1809) . This estimation method is very popular for the model fitting , especially in linear and non-linear regression . The method of least square estimator scan is produced by the minimizing sum of squares error between value and its expected value . [2] The least squares method is a combination of the parametric (F) and the non-parametric ($\hat{F}$) Distribution functions . The minimizing following equation : [6]

$$S = \sum_{i=1}^{n}\left(\hat{F}(X_i) - F(X_i)\right)^2 \quad \ldots (37)$$

Suppose that $X_1, X_2, X_3, \ldots, X_n$ be a random sample have $Ex(\beta)$ distribution with the sample size n . The procedure attempts to minimize the following β will get as :

$$S(\beta) = \sum_{i=1}^{n}\left(\hat{F}(X_i) - (1 - e^{-\beta x})\right)^2 \quad \ldots (38)$$

To obtain the formula of $F(X_i)$; use the equation (6) :

$$F(X_i) = 1 - e^{-\beta x_i}$$

$$-ln\left(1 - F(X_i)\right) = \beta x_i \quad \ldots (39)$$

On the other hand , since $\hat{F}(X_i)$ is unknown , it better to use $\hat{F}(X_{(i)})$ as follows $\hat{F}\left(X_{(i)}\right) = P_i$ and P_i is the plotting position Where

$$P_i = \frac{i}{n+1} \;;\; i = 1,2,\ldots,n \quad \ldots (40)$$

Here $X_{(i)}$ is the i: th order statistics of the random sample of the size n from ED. Hence for the ED , to obtain the LS estimates $\hat{\beta}$ of the parameter β can be define following the function from equation (38) :

$$S(\beta) = \sum_{i=1}^{n}(q_i - \beta X_{(i)})^2 \quad \ldots (41)$$

Where $q_i = -ln\left(1 - \hat{F}\left(X_{(i)}\right)\right) = -ln(1 - p_i)$

By taking the derivative equation (41) with respect to the parameter β and equating result to the zero :

$$\frac{\partial S(\beta)}{\beta} = \sum_{i=1}^{n} 2\left(q_i - \beta X_{(i)}\right)\left(-X_{(i)}\right)$$

$$\rightarrow \quad -\sum_{i=1}^{n} q_i X_{(i)} + \beta \sum_{i=1}^{n} X_{(i)}^2 = 0 \qquad \dots(42)$$

Then the least squares estimator of β ; says $\hat{\beta}_{(LS)}$, will get as:

$$\hat{\beta}_{(LS)} = \frac{\sum_{i=1}^{n} q_i X_{(i)}}{\sum_{i=1}^{n} {X_{(i)}}^2} \qquad \dots(43)$$

In the same way, the least squares estimator of unknown parameter μ; says $\hat{\mu}_{(LS)}$; is:

$$\hat{\mu}_{(LS)} = \frac{\sum_{j=1}^{m} q_j Y_{(j)}}{\sum_{j=1}^{m} {Y_{(j)}}^2} \qquad \dots(44)$$

Where $\hat{G}\left(y_{(j)}\right) = \frac{j}{m+1}$; $j = 1,2,\dots,m$ and

$$q_j = -ln\left(1 - \hat{G}\left(Y_{(j)}\right)\right) = -ln\left(1 - P_j\right)$$

Now , by using the same way , the last squares estimator of the unknown scale parameters $(\beta_1, \beta_2, \beta_3, \dots, \beta_N)$ and $(\mu_1, \mu_2, \mu_3, \dots, \mu_N)$ are:

$$\hat{\beta}_{\delta(LS)} = \frac{\sum_{i_\delta=1}^{n_\delta} q_{i_\delta} X_{\delta_{(i_\delta)}}}{\sum_{i_\delta=1}^{n_\delta} {X_{\delta_{(i_\delta)}}}^2} \quad , \delta = 1,2,3, \dots, N \qquad \dots(45)$$

and

$$\hat{\mu}_{\delta(LS)} = \frac{\sum_{j_\delta=1}^{m_\delta} q_{j_\delta} Y_{\delta_{(j_\delta)}}}{\sum_{j_\delta=1}^{m_\delta} {Y_{\delta_{(j_\delta)}}}^2} \quad , \delta = 1,2,3, \dots, N \qquad \dots(46)$$

Substitution (45) and (46) in (21) , the last squares estimator for reliability R_{Ex} says $\hat{R}_{Ex(LS)}$; approximately will be as:

$$\hat{R}_{Ex(LS)} = \hat{R}_{1(LS)} + \hat{R}_{2(LS)} + \hat{R}_{3(LS)} + \cdots + \hat{R}_{(N+1)(LS)}$$

$$= \left(\prod_{r=1}^{N} \frac{\hat{\mu}_{r(LS)}}{\hat{\beta}_{r(LS)} + \hat{\mu}_{r(LS)}}\right)\left(1 + \sum_{r=1}^{N} \frac{\hat{\beta}_{r(LS)}\left(\hat{\mu}_{r(LS)} + \hat{\beta}_{r(LS)}\right)}{\left(\hat{\mu}_{r(LS)} + \frac{k}{m}\hat{\beta}_{r(LS)}\right)\left[\left(1 + \frac{k}{m}\right)\hat{\beta}_{r(LS)} + \hat{\mu}_{r(LS)}\right]}\right)$$

$$= \left(\frac{\hat{\mu}_{(LS)}}{\hat{\beta}_{(LS)} + \hat{\mu}_{(LS)}}\right)^N \left(1 + \frac{N\hat{\beta}_{(LS)}\left(\hat{\mu}_{(LS)} + \hat{\beta}_{(LS)}\right)}{\left(\hat{\mu}_{(LS)} + \frac{k}{m}\hat{\beta}_{(LS)}\right)\left[\left(1 + \frac{k}{m}\right)\hat{\beta}_{(LS)} + \hat{\mu}_{(LS)}\right]}\right) \qquad \dots(47)$$

3.4 Weighted Least Squares Estimation Method (WLS):

The method weighted least squares extend the method least squares procedure to case where the sample data have different variance . By other words , some the samples have more error or less influence than others . This method reflects the behavior of random errors in the model and it can be used with the functions that are either linear or nonlinear in parameters . It works by incorporating extra nonnegative weights or constants associated with all data point into the fitting criterion . The size of weight shows the precision of the information contained in associated observation [14]. The method of weighted last squares can be used in minimizing the following equation :[1]

$$Q = \sum_{i=1}^{n} W_i \left(\hat{F}(X_i) - F(X_i)\right)^2 \qquad \dots(48)$$

Where $W_i = \frac{1}{Var[F(X_{(i)})]} = \frac{(n+1)^2(n+2)}{i(n-i+1)}$, $i = 1,2,\dots,n$ $\qquad \dots(49)$

Let a random sample $(X_1, X_2, X_4, \dots, X_n)$ size n take from have $Ex(\beta)$ distribution . The procedure attempts to minimize the following function with respect to β will get as :

$$Q(\beta) = \sum_{i=1}^{n} W_i \left(\hat{F}(X_i) - (1 - e^{-\beta x_i})\right)^2 \qquad \dots(50)$$

As steps in equations (39) and (41) will get as :

$$Q(\beta) = \sum_{i=1}^{n} W_i \left(q_i - \beta x_{(i)}\right)^2 \qquad \dots(51)$$

By taking partial derivative to the equation (51) with respect to β , and equating result to the zero we obtain :

$$\frac{\partial Q(\beta)}{\partial \beta} = \sum_{i=1}^{n} 2W_i\left(q_i - \beta X_{(i)}\right)\left(-X_{(i)}\right)$$

$$\rightarrow -\sum_{i=1}^{n} W_i q_i X_{(i)} + \hat{\beta} \sum_{i=1}^{n} W_i X_{(i)}^2 = 0 \qquad \dots(52)$$

The weighted least square estimator of β ; says $\hat{\beta}_{(WLS)}$:

$$\hat{\beta}_{(WLS)} = \frac{\sum_{i=1}^{n} W_i q_i X_{(i)}}{\sum_{i=1}^{n} W_i X_{(i)}^2} \qquad \dots(53)$$

In the same technique , the weighted least squares estimator of unknown scale parameter μ ; says $\hat{\mu}$(WLS) ; is :

$$\hat{\mu}_{(WLS)} = \frac{\sum_{i=1}^{m} W_i q_i Y_{(j)}}{\sum_{i=1}^{m} W_i Y_{(i)}^2} \qquad \dots(54)$$

$$\text{Where} \quad W_j = \frac{1}{Var\left[G\left(Y_{(j)}\right)\right]} = \frac{(m+1)^2(m+2)}{j(m-j+1)}$$

$, j = 1,2, \dots., m$

Now , by using the same way , the weighted least squares estimators of the unknown scale parameters $(\beta_1, \beta_2, \beta_3, \dots, \beta_N)$ and $(\mu_1, \mu_2, \mu_3, \dots, \mu_N)$ are:

$$\hat{\beta}_{\delta(WLS)} = \frac{\sum_{i_\delta=1}^{n_\delta} W_{i_\delta} q_{i_\delta} X_{\delta(i_\delta)}}{\sum_{i_\delta=1}^{n_\delta} W_{i_\delta} X_{\delta(i_\delta)}^2} \quad , \delta = 1,2,3,\dots,N \qquad \dots(55)$$

and

$$\hat{\mu}_{\delta(WLS)} = \frac{\sum_{i_\delta=1}^{n_\delta} W_{i_\delta} q_{i_\delta} Y_{\delta(i_\delta)}}{\sum_{i_\delta=1}^{n_\delta} W_{i_\delta} Y_{\delta\ (i_\delta)}^{2\alpha}} \quad , \delta = 1,2,3, \dots, N \qquad \dots (56)$$

Substitution (55) and (56) in (21) , the weighted least squares estimator for reliability R_{Ex} ; says $\hat{R}_{Ex(WLS)}$; approximately will be as:

$$\hat{R}_{Ex(WLS)} = \hat{R}_{1(WLS)} + \hat{R}_{2(WLS)} + \hat{R}_{3(WLS)} + \cdots + \hat{R}_{(N+1)(WLS)}$$

$$= \left(\prod_{r=1}^{N} \frac{\hat{\mu}_{r(WLS)}}{\hat{\beta}_{r(WLS)} + \hat{\mu}_{r(WLS)}}\right)\left(1 + \sum_{r=1}^{N} \frac{\hat{\beta}_{r(WLS)}\left(\hat{\mu}_{r(WLS)} + \hat{\beta}_{r(WLS)}\right)}{\left(\hat{\mu}_{r(WLS)} + \frac{k}{m}\hat{\beta}_{r(WLS)}\right)\left[\left(1 + \frac{k}{m}\right)\hat{\beta}_{r(WLS)} + \hat{\mu}_{r(WLS)}\right]}\right)$$

$$= \left(\frac{\hat{\mu}_{(WLS)}}{\hat{\beta}_{(WLS)} + \hat{\mu}_{(WLS)}}\right)^{N}\left(1 + \frac{N\hat{\beta}_{(WLS)}\left(\hat{\mu}_{(WLS)} + \hat{\beta}_{(WLS)}\right)}{\left(\hat{\mu}_{(WLS)} + \frac{k}{m}\hat{\beta}_{(WLS)}\right)\left[\left(1 + \frac{k}{m}\right)\hat{\beta}_{(WLS)} + \hat{\mu}_{(WLS)}\right]}\right)$$

... (57)

3.5 Regression Estimation Method (Rg):

Regression is one of the important procedures that uses supplementary information to construct estimators with a good efficiency . Regression is conceptually the simple method for examining functional relations among variables . The relations is expressed in form of an equation or the model connecting the response variable "Y" and one "X" or more expository variables . The simple true relations can be approximated by the standard regression equation :[16]

$$z_i = a + b\mu_i + e_i \qquad \dots (58)$$

Where (z_i) is the dependent variable , (u_i) is the independent variable and (e_i) is error random variable independent.

Assume that $X_1, X_2, \dots, X_n$ random samples have $Ex(\beta)$ with the sample size n .

Taking the natural logarithm to CDF [17] , obtain by equation (6):

$$F(X_i) = 1 - e^{-\beta x_i}$$

$$(1 - F(X_i))^{-1} = e^{\beta x_i}$$

$$Ln[(1 - F(X_i))^{-1}] = \beta X_i$$

Estimating $F(X_{(i)})$ by P_i in equation (40)

$$Ln[(1 - P_i)^{-1}] = \beta X_{(i)} \qquad \dots (59)$$

Comparing the equation (59) with equation (58),we get :

$$z_i = Ln[(1 - P_i)^{-1}], a = 0, b = \beta, u_i = X_{(i)}$$

Where ; $i = 1,2, \dots, n$ ...(60)

Where b can be estimated by the minimizing summation of the squared error with respect to b ,then we get :

$$\hat{b} = \frac{n\sum_{i=1}^{n} z_i u_i - \sum_{i=1}^{n} z_i \sum_{i=1}^{n} u_i}{n\sum_{i=1}^{n}(u_i)^2 - \left(\sum_{i=1}^{n} u_i\right)^2} \qquad \dots (61)$$

By substation (60) in (61) , the estimator for β ; says $\hat{\beta}_{(Rg)}$; is:

$$\hat{\beta}_{(Rg)} = \frac{n\sum_{i=1}^{n} X_{(i)}\ ln[(1-P_i)^{-1}] - \sum_{i=1}^{n} X_{(i)}\ \sum_{i=1}^{n} Ln[(1-P_i)^{-1}]}{n\sum_{i=1}^{n}[X_{(i)}]^2 - [\sum_{i=1}^{n} X_{(i)}]^2} \quad \dots (62)$$

In the same way , the regression estimator of unknown scale parameter μ ; says $\hat{\mu}_{(Rg)}$; is:

$$\hat{\mu}_{(Rg)} = \frac{m\sum_{j=1}^{m} Y_{(j)}\ ln\left[\left(1-P_j\right)^{-1}\right] - \sum_{j=1}^{m} Y_{(j)}\ \sum_{j=1}^{m} Ln\left[\left(1-P_j\right)^{-1}\right]}{m\sum_{j=1}^{m}[Y_{(j)}]^2 - [\sum_{j=1}^{m} Y_{(j)}]^2} \quad \dots (63)$$

As in equation (60) where $z_j = ln\left[\left(1-P_j\right)^{-1}\right]$

$$; j = 1,2,\dots,m \ , a = 0 \ , b = \theta, u_j = {y_{(j)}}^{\alpha}$$

Now , by using the same way above , the regression estimators of the unknown scale parameters $(\beta_1, \beta_2, \beta_3, \dots, \beta_N)$ and $(\mu_1, \mu_2, \mu_3, \dots, \mu_N)$ are :

$$\hat{\beta}_{\delta(Rg)} = \frac{n_\delta \sum_{i_\delta=1}^{n_\delta} X_{\delta_{(i_\delta)}}\ ln\left[\left(1-P_{i_\delta}\right)^{-1}\right] - \sum_{i_\delta=1}^{n_\delta} X_{\delta_{(i_\delta)}}\ \sum_{i_\delta=1}^{n_\delta} Ln\left[\left(1-P_{i_\delta}\right)^{-1}\right]}{n_\delta \sum_{i_\delta=1}^{n_\delta}\left[X_{\delta_{(i_\delta)}}\right]^2 - \left[\sum_{i_\delta=1}^{n_\delta} X_{\delta_{(i_\delta)}}\right]^2}$$

$, \delta = 1,2,3, \dots, N$ ... (64)

and

$$\hat{\mu}_{\delta(Rg)} = \frac{m_\delta \sum_{j_\delta=1}^{m_\delta} Y_{\delta_{(j_\delta)}}\ ln\left[\left(1-P_{j_\delta}\right)^{-1}\right] - \sum_{j_\delta=1}^{m_\delta} Y_{\delta_{(j_\delta)}}\ \sum_{j_\delta=1}^{m_\delta} Ln\left[\left(1-P_{j_\delta}\right)^{-1}\right]}{m_\delta \sum_{j_\delta=1}^{m_\delta}\left[Y_{\delta_{(j_\delta)}}\right]^2 - \left[\sum_{j_\delta=1}^{m_\delta} Y_{\delta_{(j_\delta)}}\right]^2}$$

$, \delta = 1,2,3, \dots, N$ ... (65)

Substitution (64) and (65) in (21) , the regression estimator for reliability R_{Ex} ; says $\hat{R}_{Ex(Rg)}$; approximately will be as :

$$\hat{R}_{Ex(Rg)} = \hat{R}_{1(Rg)} + \hat{R}_{2(Rg)} + \hat{R}_{3(Rg)} + \dots + \hat{R}_{(N+1)(Rg)}$$

$$= \left(\prod_{r=1}^{N} \frac{\hat{\mu}_{r(Rg)}}{\hat{\beta}_{r(Rg)} + \hat{\mu}_{r(Rg)}}\right)\left(1 + \sum_{r=1}^{N} \frac{\hat{\beta}_{r(Rg)}\left(\hat{\mu}_{r(Rg)} + \hat{\beta}_{r(Rg)}\right)}{\left(\hat{\mu}_{r(Rg)} + \frac{k}{m}\hat{\beta}_{r(Rg)}\right)\left[\left(1+\frac{k}{m}\right)\hat{\beta}_{r(Rg)} + \hat{\mu}_{r(Rg)}\right]}\right)$$

$$= \left(\frac{\hat{\mu}_{(Rg)}}{\hat{\beta}_{(Rg)} + \hat{\mu}_{(Rg)}}\right)^N \left(1 + \frac{N\hat{\beta}_{(Rg)}\left(\hat{\mu}_{(Rg)} + \hat{\beta}_{(Rg)}\right)}{\left(\hat{\mu}_{(Rg)} + \frac{k}{m}\hat{\beta}_{(Rg)}\right)\left[\left(1+\frac{k}{m}\right)\hat{\beta}_{(Rg)} + \hat{\mu}_{(Rg)}\right]}\right) \quad \dots (66)$$

3.6 Percentile Estimation Method (Pr):

The method was originally discovered by Kao (1958 – 1959) . In case of Exponential distribution , let a random sample X_i ; $i = 1,2,3, \dots, n$ with size n have $Ex(\beta)$, it is possible to use this method to obtain the estimator unknown scale parameter β , which is obtain from the CDF , defined in equation (6):[7]

$$F(X_i) = 1 - e^{-\beta x_i}$$

$$ln\left(1 - F(X_i)\right) = -\beta x_i$$

$$X_i = \frac{-ln(1-F(X_i))}{\beta} \qquad \dots(67)$$

If $P_i\,; i = 1,2,\dots,n$ put the plotting position instead of $F(X_{(i)};\beta)$. Can be obtained by minimizing

$$\sum_{i=1}^{n}\left[X_{(i)} - \left(\frac{-\ln(1-p_i)}{\beta}\right)\right]^2 \qquad \dots(68)$$

By taking partial derivative to the (68) with respect to β , and equating the result to zero we obtain:

$$\sum_{i=1}^{n} 2\left[(X_{(i)}) - (\hat{\beta})^{-1}(-ln(1-p_i))\right](\hat{\beta})^{-2}(-ln(1-p_i)) = 0$$

$$\rightarrow \sum_{i=1}^{n}(X_{(i)})(-\ln(1-p_i)) - (\hat{\beta})^{-1}[\ln(1-p_i)]^2 = 0$$

$$\rightarrow \sum_{i=1}^{n} X_{(i)}\ln(1-p_i) + (\hat{\beta})^{-1}\sum_{i=1}^{n}[\ln(1-p_i)]^2 = 0$$

The percentile estimator of β ; says $\hat{\beta}$ becomes :

$$\hat{\beta}_{(pr)} = \frac{-\sum_{i=1}^{n}[ln(1-p_i)]^2}{\sum_{i=1}^{n} X_{(i)}\, ln(1-p_i)} \qquad \dots(69)$$

In the name way above , the percentile estimator of the unknoun parameter $\hat{\beta}$; says $\hat{\mu}$; is:

$$\hat{\mu}_{(pr)} = \frac{-\sum_{j=1}^{m}[ln(1-p_j)]^2}{\sum_{j=1}^{m} Y_{(j)}\, ln(1-p_j)} \qquad \dots(70)$$

Now , by using the same manner , the percentile estimators of the unknown scale parametor $(\beta_1,\beta_2,\beta_3,\dots,\beta_N)$ and $(\hat{\beta}_1,\hat{\beta}_2,\hat{\beta}_3,\dots,\hat{\beta}_N)$ are:

$$\hat{\beta}_{\delta(pr)} = \frac{-\sum_{i_\delta=1}^{n_\delta}[ln(1-p_{i_\delta})]^2}{\sum_{i_\delta=1}^{n_\delta}(X_{(i_\delta)})ln(1-p_{i_\delta})} \quad ,\delta = 1,2,3,\dots,N \qquad \dots(71)$$

And

$$\hat{\mu}_{\delta(pr)} = \frac{-\sum_{j_\delta=1}^{m_\delta}[ln(1-p_{j_\delta})]^2}{\sum_{j_\delta=1}^{m_\delta}(Y_{(j_\delta)})ln(1-p_{j_\delta})} \quad ,\delta = 1,2,3,\dots,N \qquad \dots(72)$$

Substituting (71) and (72) in (21) , the percentile estimator for reliability R ; $\hat{R}_{(pi)}$; invariability will be as:

$$\hat{R}_{Ex(pr)} = \hat{R}_{1(pr)} + \hat{R}_{2(pr)} + \hat{R}_{3(pr)} + \cdots + \hat{R}_{(N+1)(pr)}$$

$$= \left(\prod_{r=1}^{N}\frac{\hat{\mu}_{r(pr)}}{\hat{\beta}_{r(pr)} + \hat{\mu}_{r(pr)}}\right)\left(1 + \sum_{r=1}^{N}\frac{\hat{\beta}_{r(pr)}(\hat{\mu}_{r(pr)} + \hat{\beta}_{r(pr)})}{\left(\hat{\mu}_{r(pr)} + \frac{k}{m}\hat{\beta}_{r(pr)}\right)\left[\left(1+\frac{k}{m}\right)\hat{\beta}_{r(pr)} + \hat{\mu}_{r(pr)}\right]}\right)$$

$$= \left(\frac{\hat{\mu}_{(Pr)}}{\hat{\beta}_{(Pr)} + \hat{\mu}_{(Pr)}}\right)^{N} \left(1 + \frac{N\hat{\beta}_{(Pr)}\left(\hat{\mu}_{(Pr)} + \hat{\beta}_{(Pr)}\right)}{\left(\hat{\mu}_{(Pr)} + \frac{k}{m}\hat{\beta}_{(Pr)}\right)\left[\left(1 + \frac{k}{m}\right)\hat{\beta}_{(Pr)} + \hat{\mu}_{(Pr)}\right]}\right) \qquad \dots(73)$$

3.7 Pitman Method:

Let $X_1, X_2, \dots, X_n$ be a random sample of n observations from a population whose p.d.f is $f(x,\beta)$; where $\beta > 0$ is a scale parameter and $x_i > 0$. if $\hat{\beta} = g(X_1, X_2, \dots, X_n)$ is the estimator of the scale parameter β ,then $\hat{\beta}$ should be as follows :[19]

$$f(x,\beta) = \beta e^{-\beta x}$$

$$L(X_1, X_2, \dots, X_n, \beta) = f(X_1,\beta) f(X_2,\beta) \dots f(X_n,\beta)$$

$$= \prod_{i=1}^{n} f(X_i,\beta) \qquad \dots(74)$$

$$L(X_1, X_2, \dots, X_n, \beta) = \prod_{i=1}^{n} \left[\beta e^{-\beta x_i}\right]$$

$$\mathrm{L}(X_1, X_2, \dots, X_n, \beta) = \beta^n e^{-\beta \sum_{i=1}^{n} x_i} \qquad \dots(75)$$

$$\hat{\beta} = \frac{\int_0^\infty \frac{1}{\beta^2} \mathrm{L}(X_1, X_2, \dots, X_n, \beta) d\beta}{\int_0^\infty \frac{1}{\beta^3} \mathrm{L}(X_1, X_2, \dots, X_n, \beta) d\beta} \qquad \dots(76)$$

$$= \frac{\int_0^\infty \frac{1}{\beta^2} \beta^n e^{-\beta \sum_{i=1}^{n} x_i} d\beta}{\int_0^\infty \frac{1}{\beta^3} \beta^n e^{-\beta \sum_{i=1}^{n} x_i} d\beta}$$

$$= \frac{\int_0^\infty \beta^{n-2} e^{-\beta \sum_{i=1}^{n} x_i} d\beta}{\int_0^\infty \beta^{n-3} e^{-\beta \sum_{i=1}^{n} x_i} d\beta}$$

Let $u = \beta \sum_{i=1}^{n} x_i \rightarrow \quad du = \sum_{i=1}^{n} x_i \, d\beta \;\; , \; d\beta = \frac{du}{\sum_{i=1}^{n} x_i} \quad and \quad \beta = \frac{u}{\sum_{i=1}^{n} x_i}$

$$\rightarrow \quad \hat{\beta} = \frac{\int_0^\infty \left(\frac{u}{\sum_{i=1}^{n} x_i}\right)^{n-2} e^{-u} \frac{du}{\sum_{i=1}^{n} x_i}}{\int_0^\infty \left(\frac{u}{\sum_{i=1}^{n} x_i}\right)^{n-3} e^{-u} \frac{du}{\sum_{i=1}^{n} x_i}}$$

$$= \frac{\frac{1}{(\sum_{i=1}^{n} x_i)^{n-1}} \int_0^\infty u^{n-2} e^{-u} du}{\frac{1}{(\sum_{i=1}^{n} x_i)^{n-2}} \int_0^\infty u^{n-3} e^{-u} du}$$

$$= \left(\frac{1}{\sum_{i=1}^{n} x_i}\right) \frac{\int_0^\infty u^{n-1-1} e^{-u} du}{\int_0^\infty u^{n-2-1} e^{-u} du} = \left(\frac{1}{\sum_{i=1}^{n} x_i}\right) \frac{\Gamma(n-1)}{\Gamma(n-2)}$$

$$= \left(\frac{1}{\sum_{i=1}^{n} x_i}\right)\frac{(n-2)!}{(n-3)!} = \left(\frac{1}{\sum_{i=1}^{n} x_i}\right)\frac{(n-2)(n-3)!}{(n-3)!} = \frac{n-2}{\sum_{i=1}^{n} x_i}$$

The Pitman estimator of β ; says $\hat{\beta}$ becomes :

$$\hat{\beta} = \frac{n-2}{\sum_{i=1}^{n} x_i} \qquad \dots(77)$$

In the name way above , the Pitman estimator of the unknown parametor $\hat{\beta}$; says $\hat{\mu}$; is:

$$\hat{\mu} = \frac{m-2}{\sum_{j=1}^{m} y_j} \qquad \dots(78)$$

Now , by using the same manner , the Pitman estimators of the unknown scale parameter $(\beta_1, \beta_2, \beta_3, \dots, \beta_N)$ and $(\hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3, \dots, \hat{\beta}_N)$ are :

$$\hat{\beta}_{\delta(pi)} = \frac{n_\delta - 2}{\sum_{i_\delta=1}^{n_\delta} x_{i_\delta}} \quad , \delta = 1,2,3, \dots, N \qquad \dots(79)$$

And

$$\hat{\mu}_{\delta(pi)} = \frac{m_\delta - 2}{\sum_{j_\delta=1}^{m_\delta} y_{j_\delta}} \quad , \delta = 1,2,3, \dots, N \qquad \dots(80)$$

Substituting (79) and (80) in (21) , the Pitman estimator for reliability R ; $\hat{R}_{(pi)}$; invariability will be as:

$$\hat{R}_{Ex(pi)} = \hat{R}_{1(pi)} + \hat{R}_{2(pi)} + \hat{R}_{3(pi)} + \cdots + \hat{R}_{(N+1)(pi)}$$

$$= \left(\prod_{r=1}^{N} \frac{\hat{\mu}_{r(pi)}}{\hat{\beta}_{r(pi)} + \hat{\mu}_{r(pi)}}\right)\left(1 + \sum_{r=1}^{N} \frac{\hat{\beta}_{r(pi)}\left(\hat{\mu}_{r(pi)} + \hat{\beta}_{r(pi)}\right)}{\left(\hat{\mu}_{r(pi)} + \frac{k}{m}\hat{\beta}_{r(pi)}\right)\left[\left(1 + \frac{k}{m}\right)\hat{\beta}_{r(pi)} + \hat{\mu}_{r(pi)}\right]}\right)$$

$$= \left(\frac{\hat{\mu}_{(Pi)}}{\hat{\beta}_{(Pi)} + \hat{\mu}_{(Pi)}}\right)^N \left(1 + \frac{N\hat{\beta}_{(Pi)}\left(\hat{\mu}_{(Pi)} + \hat{\beta}_{(Pi)}\right)}{\left(\hat{\mu}_{(Pi)} + \frac{k}{m}\hat{\beta}_{(Pi)}\right)\left[\left(1 + \frac{k}{m}\right)\hat{\beta}_{(Pi)} + \hat{\mu}_{(Pi)}\right]}\right) \qquad \dots(81)$$

4. The experimental study:

We simulate the outputs of all three estimating methods by using MSE . Study of simulation is replicated several times (500) so that the samples of three sizes (small , moderate and large) are independently collected.

4.1 Algorithm of Simulation:

The simulation algorithms are written for estimating R using MATLAB program , according to the following steps:

1- The random sample $(x_{11}, x_{12}, \dots, x_{1r_1})$, $(x_{21}, x_{22}, \dots, x_{2r_2})$,

$(x_{31}, x_{32}, \dots, x_{3r_3})$, and $(y_{11}, y_{12}, \dots, y_{1v_1})$,

$(y_{21}, y_{22}, \dots, y_{2v_2})$, $(y_{31}, y_{32}, \dots, y_{3v_3})$

of sizes $(r_1, r_2, r_3, v_1, v_2, v_3) = (15,15,15,15,15,15)$

, $(45,45,45,45,45,45)$ *and* $(95,95,95,95,95,95)$

are generated from Exponentiol distribution.

2- Selected the values of parameters for 7 experiments $(\beta_1,\beta_2,\beta_3,\dots,\beta_N,\mu_1,\mu_2,\mu_3,\dots,\mu_N)$ in the table (1) :

Table (1) : Values of parameters and Reliability , $N = 10$.

Experiment	k	m	β	μ	R
1	2	0.5	0.2	1.5	0.4552
2	2	0.5	0.3	1.5	0.2692
3	2	0.5	0.1	1.5	0.7453
4	2	0.5	0.2	2	0.5875
5	2	0.5	0.2	1	0.2692
6	2.5	0.5	0.2	1.5	0.4301
7	1.5	0.5	0.2	1.5	0.4874

3-Parameters $\beta_1,\beta_2,\beta_3,\dots,\beta_N,\mu_1,\mu_2,\mu_3,\dots,\mu_N$ were estimated (ML,Mo , LS , WLS , Rg and Pr) in equations: (28),(29),(34),(35),(45),(46),(55),(56),(64),(65),(71),(72) , (79) and (80) respectively.

4- R was estimated in equations: (30),(36),(47),(57),(66) ,(73) and (81) .

5- Calculate the mean by Mean $= \frac{\sum_{i=1}^{L} R_i}{L}$

6- The last stage is to use the " Mean square Error " to assess the results of the seven estimation methods:

$$MSE(\hat{R}) = \frac{1}{L}\sum_{i=1}^{L}(\hat{R}_i - R)^2$$

Simulation Results:

After applying the previous steps of R for sample size $(r_1,r_2,r_3,\dots,r_N,v_1,v_2,v_3,\dots,v_N)$: $(15,15,15,\dots,15,15,15,15,\dots,15)$, $(95,95,95,\dots,95,95,95,95,\dots,95)$ and $(45,45,45,\dots,45,45,45,45,\dots,45)$

Table (2) : Values Mean and MSE for 7 experiments, $N = 10$.

Exp	Simple size	Criterion	ML	MO	LS	WLS	Rg	Pr	Pi	Best
	(10,10,10,…,10,10,10,10,…,10)	Mean	0.4551	0.4551	0.4526	0.4522	0.4496	0.4528	0.4551	
		MSE	0.0356	0.0356	0.0384	0.0415	0.0493	0.0372	0.0356	ML , MO , Pi
1	(50,50,50,…,50,50,50,50,…,50)	Mean	0.4585	0.4585	0.4594	0.4616	0.4603	0.4595	0.4585	

		MSE	0.0083	0.0083	0.0102	0.0170	0.0153	0.0093	0.0083	ML, MO, Pi
	(90,90,90,…,90,90,90,90,…,90)	Mean	0.4567	0.4567	0.4557	0.4528	0.4545	0.4560	0.4567	
		MSE	0.0045	0.0045	0.0057	0.0120	0.0089	0.0053	0.0045	ML, MO, Pi
2	(10,10,10,…,10,10,10,10,…,10)	Mean	0.2904	0.2904	0.2971	0.3007	0.3112	0.2951	0.2904	
		MSE	0.0266	0.0266	0.0308	0.0347	0.0431	0.0291	0.0266	ML, MO, Pi
	(50,50,50,…,50,50,50,50,…,50)	Mean	0.2706	0.2706	0.2689	0.2705	0.2698	0.2692	0.2706	
		MSE	0.0066	0.0066	0.0077	0.0127	0.0115	0.0070	0.0066	ML, MO, Pi
	(90,90,90,…,90,90,90,90,…,90)	Mean	0.2686	0.2686	0.2681	0.2709	0.2690	0.2680	0.2686	
		MSE	0.0036	0.0036	0.0047	0.0098	0.0076	0.0044	0.0036	ML, MO, Pi
3	(10,10,10,…,10,10,10,10,…,10)	Mean	0.7175	0.7175	0.7159	0.7136	0.7054	0.7162	0.7175	
		MSE	0.0264	0.0264	0.0305	0.0337	0.0424	0.0295	0.0264	ML, MO, Pi
	(50,50,50,…,50,50,50,50,…,50)	Mean	0.7407	0.7407	0.7392	0.7350	0.7352	0.7391	0.7407	
		MSE	0.0042	0.0042	0.0051	0.0088	0.0079	0.0048	0.0042	ML,

									MO, Pi
	(90,90,90,…,90,90,90,90,…,90)	Mean	0.7417	0.7417	0.7402	0.7350	0.7372	0.7407	0.7417	
		MSE	0.0024	0.0024	0.0030	0.0063	0.0048	0.0028	0.0024	ML, MO, Pi
4	(10,10,10,…,10,10,10,10,…,10)	Mean	0.5791	0.5791	0.5793	0.5774	0.5747	0.5788	0.5791	
		MSE	0.0311	0.0311	0.0341	0.0374	0.0464	0.0332	0.0311	ML, MO, Pi
	(50,50,50,…,50,50,50,50,…,50)	Mean	0.5810	0.5810	0.5810	0.5795	0.5799	0.5814	0.5810	
		MSE	0.0071	0.0071	0.0094	0.0161	0.0146	0.0084	0.0071	ML, MO, Pi
	(90,90,90,…,90,90,90,90,…,90)	Mean	0.5847	0.5847	0.5840	0.5792	0.5820	0.5840	0.5847	
		MSE	0.0040	0.0040	0.0051	0.0103	0.0078	0.0048	0.0040	ML, MO, Pi
5	(10,10,10,…,10,10,10,10,…,10)	Mean	0.2911	0.2911	0.2959	0.2996	0.3066	0.2946	0.2911	
		MSE	0.0296	0.0296	0.0335	0.0375	0.0459	0.0322	0.0296	ML, MO, Pi
	(50,50,50,…,50,50,50,50,…,50)	Mean	0.2766	0.2766	0.2740	0.2752	0.2736	0.2740	0.2766	
		MSE	0.0064	0.0064	0.0077	0.0125	0.0115	0.0072	0.0064	ML, MO,

									Pi
	(90,90,90,…,90,90,90,90,…,90)	Mean	0.2710	0.2710	0.2738	0.2805	0.2777	0.2728	0.2710	
		MSE	0.0038	0.0038	0.0051	0.0109	0.0081	0.0047	0.0038	ML, MO, Pi
6	(10,10,10,…,10,10,10,10,…,10)	Mean	0.4308	0.4308	0.4292	0.4287	0.4314	0.4301	0.4308	
		MSE	0.0345	0.0345	0.0378	0.0408	0.0494	0.0368	0.0345	ML, MO, Pi
	(50,50,50,…,50,50,50,50,…,50)	Mean	0.4314	0.4314	0.4323	0.4338	0.4334	0.4325	0.4314	
		MSE	0.0088	0.0088	0.0103	0.0168	0.0152	0.0096	0.0088	ML, MO, Pi
	(90,90,90,…,90,90,90,90,…,90)	Mean	0.4333	0.4333	0.4346	0.4351	0.4358	0.4342	0.4333	
		MSE	0.0050	0.0050	0.0066	0.0128	0.0100	0.0062	0.0050	ML, MO, Pi
7	(10,10,10,…,10,10,10,10,…,10)	Mean	0.4938	0.4938	0.4976	0.4984	0.4998	0.4968	0.4938	
		MSE	0.0363	0.0363	0.0390	0.0419	0.0501	0.0376	0.0363	ML, MO, Pi
	(50,50,50,…,50,50,50,50,…,50)	Mean	0.4865	0.4865	0.4856	0.4863	0.4840	0.4853	0.4865	
		MSE	0.0090	0.0090	0.0115	0.0188	0.0173	0.0106	0.0090	ML, MO, Pi

	(90,90,90,…,90,90,90,90,…,90)	Mean	0.4845	0.4845	0.4841	0.4839	0.4840	0.4850	0.4845	
		MSE	0.0049	0.0049	0.0062	0.0128	0.0098	0.0057	0.0049	ML , MO , Pi

Conclusions :

This conclusions according to the simulation study results :

1. We concluded from the table (1) .

I- With increasing value of parameter β , reliability is decreasing.

II- With decreasing value of parameter β , reliability is increasing.

III- With the increasing value of parameter μ , reliability is increasing.

IV- With the decreasing value of parameter μ , reliability is decreasing.

V- With the increasing value of $\frac{K}{M}$, reliability is decreasing.

VI- With the decreasing value of $\frac{K}{M}$, reliability is increasing.

2. We concluded from the table (2) the best estimator for R is ML , MO and Pi for 7 experiments and different sample sizes.

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